Learning Goal: I will be able to use the properties of the Cartesian equation to solve problems and convert between different equations of planes.

Minds On: Different Types of Eq`ns

Action: Note and practice

Consolidation: Exit Question

Minds On

8.5 The Cartesian Equation of a Plane

Minds On

So far, we've seen all the following types of equations for lines and only some of them for planes. Fill in the table with examples of each type we have already discussed in class. Work with a partner to discuss possible examples and to make predictions about any we've not yet covered.

Equation	LINES	PLANES
Vector	$\vec{r} = \vec{r}_0 + t\vec{m}$ $\vec{m} = (a, b, c)$ $\vec{r}_0 = (x_0, y_0, z_0)$	$\vec{r} = \vec{r}_0 + s\vec{a} + t\vec{b}$ $\vec{b} = (a_2, b_2, c_2)$ $\vec{r}_0 = (x_0, y_0, z_0)$ $\vec{a} = (a_1, b_1, c_1)$
Parametric	$x = x_0 + ta$ $y = y_0 + tb$ $z = z_0 + tc$	$x = x_0 + sa_1 + ta_2$ $y = y_0 + sb_1 + tb_2$ $z = z_0 + sc_1 + tc_2$
Symmetric	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$	
Cartesian	$Ax + By + C = 0$	$Ax + By + Cz + D = 0$

Minds On

Picture it!

Using a 3D model (such as your textbook) and a plane (such a this piece of paper), you can picture a line L that is perpendicular to the plane, π. For any plane in R^3, there is only one possible line that can be drawn through the origin perpendicular to the plane. This line is called the ***normal axis*** for the plane, and the direction vector is called the ***normal*** to the plane. The direction of the normal axis is given by a vector joining the origin to any point on the normal axis. This means there are an infinite number of normals for any plane.

Minds On

Cartesian Equation of a Plane

The Cartesian (or scalar) equation of a plane in R^3 is of the form $\boldsymbol{Ax + By + Cz + D = 0}$, with normal $\vec{n} = (A, B, C)$. The normal $\vec{n}$ is a nonzero vector perpendicular to all vectors in the plane.

Minds On

Try on your own or with a partner: The point A(1, 2, 2) is a point on the plane with normal $\vec{n}$ = (-1, 2, 6). Determine the Cartesian equation of this plane.

x y z

A B C

$\hookrightarrow Ax + By + Cz + D = 0$

Let P(x, y, z) be any point on the plane.

$\vec{m} = \vec{AP} = (x-1, y-2, z-2)$

$\vec{AP} \cdot \vec{n} = 0$ * direction vector is $\perp$ to normal

$(x-1, y-2, z-2) \cdot (-1, 2, 6) = 0$

$-x + 1 + 2y - 4 + 6z - 12 = 0$

$\boxed{-x + 2y + 6z - 15 = 0}$

$(-1)(1) + (2)(2) + (6)(2) + D = 0$

$-1 + 4 + 12 + D = 0$

$15 + D = 0$

$D = -15$

$\therefore -x + 2y + 6z - 15 = 0$

or

$x - 2y - 6z + 15 = 0$

Action

Example 1: Determine the Cartesian equation of the plane containing the points A(-1, 2, 5), B(3, 2, 4), and C(-2, -3, 6).

* need a normal & a point

or

~~need a normal + direction vector~~

1. Get two direction vectors

$\vec{a} = \vec{AB} = (4, 0, -1)$

$\vec{b} = \vec{BC} = (-5, -5, 2)$

$\vec{n} = \vec{a} \times \vec{b}$

$$\begin{array}{cc} 0 & -5 \\ -1 & 2 \\ 4 & -5 \\ 0 & -5 \end{array}$$

$x = -5$

$y = -3$

$z = -20$

$\therefore \vec{n} = (-5, -3, -20)$

or

$\vec{n} = (5, 3, 20)$

$Ax + By + Cz + D = 0$

using $\vec{n}$ and pt. A

$(5)(-1) + (3)(2) + (20)(5) + D = 0$

$-5 + 6 + 100 + D = 0$

$101 + D = 0$

$D = -101$

$\boxed{\therefore\ 5x + 3y + 20z - 101 = 0}$

Action

Example 2: Determine the vector and parametric equations of the plane with Cartesian equation $x - 2y + 5z - 6 = 0$. Perform a check to verify your answer.

Method 1

Find 3 points

Let $y=1, z=1$ A

then $x-2(1)+5(1)-6=0$

$x-2+5-6=0$

$x=3$ $\therefore (3,1,1)$ is a point

Let $y=0, z=0$ B

then $x-2(0)+5(0)-6=0$

$x=6$ $\therefore (6,0,0)$ is a point

Let $y=0, z=1$ C

then $x-2(0)+5(1)-6=0$

$x=1$ $\therefore (1,0,1)$ is a point

$\vec{a} = \overrightarrow{AB} = (3,-1,-1)$

$\vec{b} = \overrightarrow{BC} = (-5,0,1)$

$$\boxed{\vec{r} = (6,0,0) + s(3,-1,-1) + t(-5,0,1)}$$

$x = 6 + 3s - 5t$

$y = -s$

$z = -s + t$

Method 2

$x - 2y + 5z - 6 = 0$

Let $y = s$ and $z = t$

Then $x - 2s + 5t - 6 = 0$

$$\left.\begin{aligned} x &= 6 + 2s - 5t \\ y &= s \\ z &= t \end{aligned}\right\} \text{parametric}$$

$\vec{r} = (6,0,0) + s(2,1,0) + t(-5,0,1)$

To check

- do $\vec{a} \times \vec{b}$
  should be $\vec{n}$ or a scalar multiple
- then plug in and solve for D

Action

Parallel and Perpendicular Planes – Picture it!

1. If π_1 and π_2 are two perpendicular planes, with normals $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, respectively, their normals are perpendicular (that is, the dot product is zero).

2. If π_1 and π_2 are two parallel planes, with normals $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, respectively, their normals are parallel (that is n_1 = k$\overrightarrow{n_2}$) for all nonzero real numbers k.

Action

Angle between Intersecting Planes

The angle, θ, between two planes, π_1 and π_2, with normals of $\vec{n_1}$ and $\vec{n_2}$, respectively, can be calculated using the formula $cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

$$\breve{n}_1 \cdot \breve{n}_2 = |\breve{n}_1||\breve{n}_2| \cos\theta$$

Action

Example 3: Determine the actute and obtuse angle between the two planes

$\pi_1: x - y - 2z + 3 = 0$ and $\pi_2: 2x + y - z + 2 = 0$.

$\vec{n}_1 = (1, -1, -2)$ $\quad$ $\vec{n}_2 = (2, 1, -1)$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$$

$$\cos\theta = \frac{(1)(2) + (-1)(1) + (-2)(-1)}{\sqrt{(1)^2 + (-1)^2 + (-2)^2}\sqrt{(2)^2 + (1)^2 + (-1)^2}}$$

$$\cos\theta = \frac{3}{6}$$

$\theta = 60^\circ$

$\therefore$ 60° is the actute angle and

120° is the obtuse angle

Consolidation

EXIT QUESTION

a) Show that the planes π_1: $2x - 3y + z - 1 = 0$ and π_2: $4x - 3y - 17z = 0$ are perpendicular. 2, -3, 2

b) Show that the planes π_3: $2x - 3y + 2z - 1 = 0$ and π_4: $2x - 3y + 2z - 3 = 0$ are parallel but not coincident.

a) If $\pi_1 \perp \pi_2$

then $\vec{n}_1 \cdot \vec{n}_2 = 0$

$\vec{n}_1 = (2, -3, 1)$

$\vec{n}_2 = (4, -3, -17)$

$\vec{n}_1 \cdot \vec{n}_2 = (2)(4) + (-3)(-3) + (1)(-17)$

$= 8 + 9 - 17$

$= 0$ ✓

$\therefore \pi_1 \perp \pi_2$

b) If $\pi_3 \parallel \pi_4$

then $\vec{n}_3 = k\vec{n}_4$, $k \in \mathbb{R}$

$\vec{n}_3 = (2, -3, 2)$

$\vec{n}_4 = (2, -3, 2)$

$\therefore \pi_3 \parallel \pi_4$

However

$D_3 \neq D_4$

$\therefore$ non-coincident